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SOLUTION OF THREE-DIMENSIONAL STEADY STATE HEAT CONDUCTION IN MULTILAYERED MATERIALS

Surface Engineering

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INTRODUCTION

Coating/substrate system and functionally graded materials (FGMs) are increasingly used in a wide range of applications to improve the performance of critical components. For such materials, the material properties vary along the depth direction. Used as coatings or interfacial zones, multilayered materials and FGMs can reduce the magnitude of residual and thermal stresses, mitigate stress concentration and increase fracture toughness [1]. A series of researches have been devoted to the simulation and investigation of heat conduction in multilayered materials. It should be pointed out that most studies have limits in either dimensions or computational efficiency; and the number of layers is also quite limited. It is of the great convenience for applications to develop explicit expressions of the frequency response functions (FRFs) to avoid the tedious numerical procedures [2]. Through the method of FRFs, the stress and displacement fields in multilayered materials have been obtained. In present paper, the closed-form FRFs for steady-state temperature field in multilayered materials are derived. It is in a recursive form, instead of numerically solving a set of equations; and there is no limit in layer number and thickness. Based on the FRFs and semi-analytical method (SAM), the solution under arbitrary heat input can be obtained.

2 Theoretical derivation

2.1 Temperature rise field in frequency domain

A half-space with *L* coatings is illustrated in Fig. 1, where the coatings are indicated by *j*=1, ..., *L*, and the half-space is labeled by *L*+1.

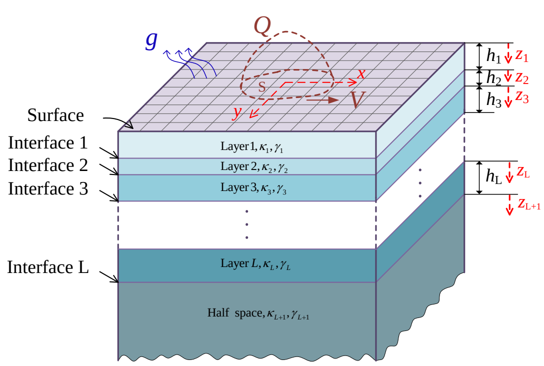

Fig. 1 Schematic of a multilayered material under unit heat input.

The partial differential equation governing heat conduction is given as follows for each layer j in its simplest form by

$$\nabla^2 T^{(j)} = -\frac{V}{\gamma_j}\frac{\partial T^{(j)}}{\partial x} \qquad \left(j = 1, \cdots, L+1\right) \tag{1}$$

As it is convenient to access the solutions of heat conduction problem for layered material in the frequency domain, here double Fourier transform (FT) is applied:

$$\frac{\partial^2 \tilde{T}^{(j)}}{\partial {z_j}^2} = \left(\omega^2 - \mathrm{i}\,\omega_x \frac{V}{\gamma_j} \right) \tilde{T}^{(j)} \tag{2}$$

where '~' stands for the double FT operation and $\omega = \sqrt{{\omega_x}^2 + {\omega_y}^2}$ with the frequency variables ω_x and ω_y corresponding to the *x* and *y* directions respectively. This is a second order linear constant coefficient differential equation with respect to z, and its general solution can be expressed in terms of exponential functions with two unknown coefficients. The general solution can be written as follows

$$\tilde{T}^{(j)} = M^{(j)} \exp\left(-r^{(j)} z_j\right) + N^{(j)} \exp\left(r^{(j)} z_j\right) \quad \left(j = 1, \cdots, L+1\right) \tag{3}$$

where $r^{(j)} = \sqrt{\omega^2 - \mathrm{i}V\omega_x / \gamma_j}$.

For the substrate (half-space), the temperature rise should be a finite-value; thus $N^{(L+1)}$ should be equal to zero and the corresponding equation in substrate is:

$$\tilde{T}^{(L+1)} = M^{(L+1)} \exp\left(-r^{(L+1)} z_{L+1}\right) \tag{4}$$

It can be seen that the total number of unknown coefficients is also 2*L*+1.

2.2 Recursive solution of the matrix equations

Substituting the temperature rise expression Eqs.(3)-(4) into the transformed boundary conditions, a linear system 2*L*+1 equations for solving 2*L*+1 unknown coefficients can be constructed. By applying double FT, the boundary conditions in frequency domain can be obtained, and can be rewritten in form of matrix equations as follows

$$\begin{bmatrix}
\kappa_1 r^{(1)} & -\kappa_1 r^{(1)} & & & & & & & 0 \\
e^{-r^{(1)}h_1} & e^{r^{(1)}h_1} & -1 & -1 & & & & & \\
-\kappa_1 r^{(1)} e^{-r^{(1)}h_1} & \kappa_1 r^{(1)} e^{r^{(1)}h_1} & \kappa_2 r^{(2)} & -\kappa_2 r^{(2)} & & & & & \\
& & & & \ddots & & & & \\
& & & & \overset{\text{Sub-matrix}}{\boxed{\begin{matrix} e^{-r^{(j)}h_j} & e^{r^{(j)}h_j} & -1 & -1 \\ -\kappa_j r^{(j)} e^{-r^{(j)}h_j} & \kappa_j r^{(j)} e^{r^{(j)}h_j} & \kappa_{j+1} r^{(j+1)} & -\kappa_{j+1} r^{(j+1)} \end{matrix}}} & & & & \\
& & & & & \ddots & & & \\
& & & & & e^{-r^{(L-1)}h_{L-1}} & e^{r^{(L-1)}h_{L-1}} & -1 & -1 \\
& & & & & -\kappa_{L-1} r^{(L-1)} e^{-r^{(L-1)}h_{L-1}} & \kappa_{L-1} r^{(L-1)} e^{r^{(L-1)}h_{L-1}} & \kappa_L r^{(L)} & -\kappa_L r^{(L)} \\
& & 0 & & & & & e^{-r^{(L)}h_L} & e^{r^{(L)}h_L} & -1 \\
& & & & & & & -\kappa_L r^{(L)} e^{-r^{(L)}h_2} & \kappa_L r^{(L)} e^{r^{(L)}h_L} & \kappa_{L+1} r^{(L+1)}
\end{bmatrix}
\begin{bmatrix} M^{(1)} \\ N^{(1)} \\ M^{(2)} \\ N^{(2)} \\ \vdots \\ M^{(j)} \\ N^{(j)} \\ M^{(j+1)} \\ N^{(j+1)} \\ \vdots \\ M^{(L-1)} \\ N^{(L-1)} \\ M^{(L)} \\ N^{(L)} \\ M^{(L+1)} \end{bmatrix}
=
\begin{bmatrix} \tilde{Q} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

For *L* layered half-space, each interface leads to two more equations for continuity of boundary conditions, which should be orderly added in the matrix equation. Then the recursive method can be applied to solve this equation. Here the bottom-up fashion is used and it starts from the interface *L*. Firstly, based on the last two equations in Eq.(5), the relation of $M^{(L)}$ and $N^{(L)}$ is obtained as

$$M^{(L)} - \frac{s^{(L)}+1}{s^{(L)}-1} e^{2r^{(L)}h_L} N^{(L)} = 0 \tag{6}$$

At interface *j*+1 is assumed to also follow the form, then combined with the sub-matrix, the relationship of $M^{(j)}$ and $N^{(j)}$ can be obtained, as shown in Eq. (7). From Eq. (7) the relationship of $M^{(j+1)}$ and $N^{(j+1)}$ can be obtained and they are also in the form of Eq.(6), which is proven to hold for every layer. Then the coefficients of the first layer can be solved through Eq.(8), and all the coefficients can be obtained with a top-bottom fashion.

$$\begin{bmatrix}
e^{-r^{(j)}h_j} & e^{r^{(j)}h_j} & -1 & -1 \\
-G^{(j)} e^{-r^{(j)}h_j} & G^{(j)} e^{r^{(j)}h_j} & 1 & -1 \\
0 & 0 & 1 & -\dfrac{s^{(j+1)}+1}{s^{(j+1)}-1} e^{2r^{(j+1)}h_{j+1}}
\end{bmatrix}
\begin{bmatrix} M^{(j)} \\ N^{(j)} \\ M^{(j+1)} \\ N^{(j+1)} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \kappa_1 r^{(1)} & -\kappa_1 r^{(1)} \\ 1 & -\dfrac{s^{(1)}+1}{s^{(1)}-1} e^{2r^{(1)}h_1} \end{bmatrix} \begin{bmatrix} M^{(1)} \\ N^{(1)} \end{bmatrix} = \begin{bmatrix} \tilde{Q} \\ 0 \end{bmatrix} \tag{8}$$

3 Numerical results and discussions

Firstly the model is verified through the comparison with FEM for the case of four-coatings on a half-space (L=4, κ_j=1, 0.2, 1, 0.2, 1, j=1,…,5) under rectangular unit heat source, as shown in Fig.2. Through the comparison the present method shows high accuracy and efficiency.

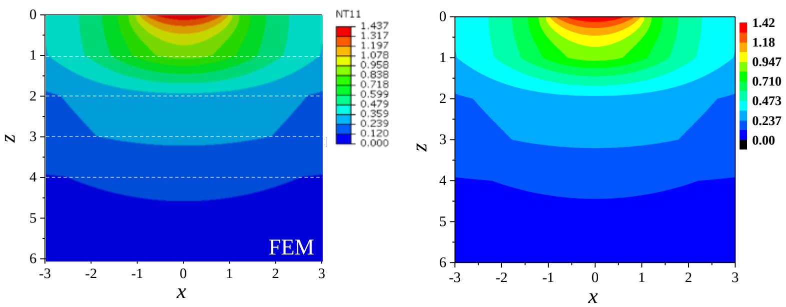

Fig. 2 Contour plots of temperature rise on y=0 plane from FEM and the present method.

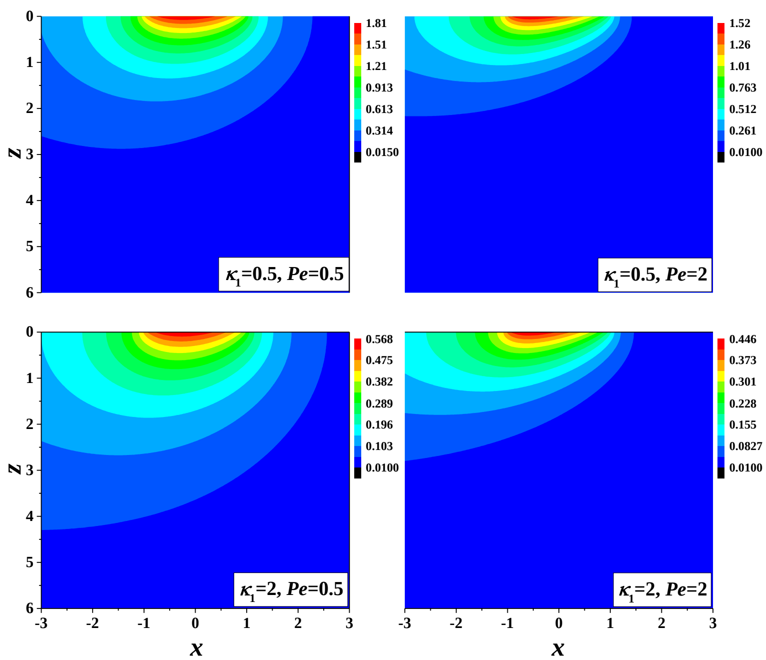

Fig. 3 Temperature rise on y=0 plane for different thermal conductivities and velocities of moving heat source.

Then the exponentially varying thermal conductivity coefficient in FGM is studied as shown in Eq.(9).

$$\kappa(z) = \kappa_1 e^{\beta z}, \quad \beta = \ln\left(\kappa_{L+1} / \kappa_1\right) / h_c \tag{9}$$

The case that the heat source moves along x-axis is studied and the Péclet number is adopted as $Pe=V^*l/\gamma$, the thermal diffusivity γ in all the layers are treated as same to unit. Then different Pe and κ_1 are applied and the result are shown in Fig.3. Firstly, for Pe=0.5 (the first column of Fig.3) the heat diffuses farther obviously as the speed is low and there is sufficient time to heat diffusion, and the maximum temperature rise decreases a lot when κ_1=2 compared that from κ_1=0.5. Then by comparing the two plots in the first or second row, it can also be found that with increase of Pe, the maximum temperature rise decreases for certain κ_1 since that the fresh material passes through the heated zone fast and the accumulative effect of heat is weakened.

4 Conclusion

Analytic FRFs for steady-state three-dimensional temperature rise solution of materials with arbitrary layers of coatings are explicitly derived for the first time. Based on the obtained FRFs of temperature rise, a semi-analytical method (SAM) is developed, which can be used to solve the temperature distribution in multilayered material under arbitrary distributed heat source. The temperature rise of materials with various number and properties of layers are solved and compared with FEM, which show excellent agreements. The temperature solution in multilayered material with exponentially varying thermal conductivity of layers along the thickness is studied. The multilayered materials with low conduction capacity present high temperature rise.

ACKNOWLEDGMENTS

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REFERENCES

1.Suresh, S. and A. Mortensen, Fundamentals of functionally graded materials: processing and thermomechanical behaviour of graded metals and metal-ceramic composites. Ugeskrift for Laeger, 1997. 135(41): p. 2203-6.

2.Yu, C., Z. Wang, and Q.J. Wang, Analytical frequency response functions for contact of multilayered materials. Mechanics of Materials, 2014. 76: p. 102-120.

KEYWORDS

Multilayered materials; heat conduction; frequency response functions; Moving heat source